Quantum Noise Theory

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Date: 6/8/2025

Outline

- Noise Definition
- Noisy Intermediate-Scale Quantum Era
- Quantum Background
- Types of Noise
- Noise Characterization
- Noise Mitigation

What is noise in quantum computing?

Any deviation from the ideal intended quantum state of a quantum circuit.

Noisy Intermediate Scale Quantum Era

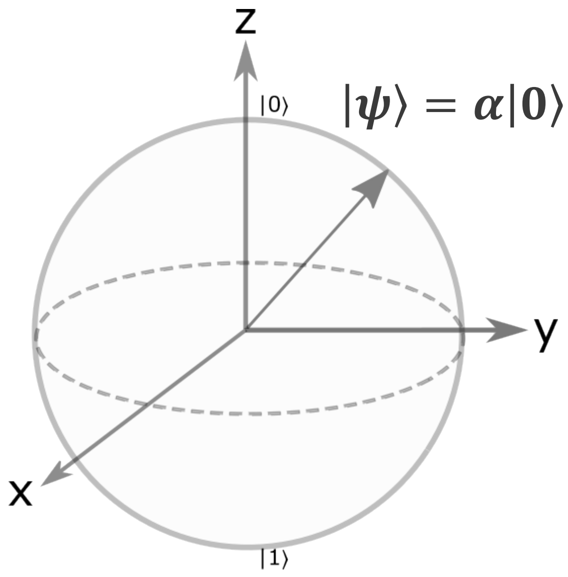
- **NISQ** termed by John Preskill in 2018.
- **Noisy** refers to limited uncontrollable noise.
- **Intermediate-Scale** refers to limited available number of qubits (ranging from 50 to a few hundred qubits)

Quantum Background

Quantum Bits

- A classical bit can be either 0 or 1.
- A quantum bit (qubit) can be any linear combination of both 0 and 1.

Superposition



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Qubit Representation

- "$\langle | | \rangle$" is the Dirac notation also known as the bra-ket notation.
- "$\langle |$" is a row vector (also known as the bra).
- "$| \rangle$" is the column vector (also known as the ket).
- Qubit at state $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or state $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- In the bra notation $\langle 0| = [1 \quad 0]$ or state $\langle 1| = [0 \quad 1]$
- Qubit can be in a linear combination of both states (**Superposition !!**)
  - $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers
  - $|\psi\rangle = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix}$
  - $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

    α → How much the qubit is in state |0>

    β → How much the qubit is in state |1>

Pure and Mixed States

- Pure state:
  - can be represented in a vector notation.

$$|\psi_{pure}\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{2^n-1} \end{bmatrix}$$

  - Lies on the surface of the Bloch sphere
  - can also be represented as a density matrix

$$\rho = |\psi_{pure}\rangle\langle\psi_{pure}|$$

- Mixed State:
  - probability distribution of several pure states (not superposition!!!)
  - Lies inside the Bloch sphere
  - can only be represented using a density matrix

    $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where p_i represents the probability to be in pure state $|\psi_i\rangle$

Quantum Gates

- Any unitary operator that can be applied to qubits to transform its state.
- Can be performed on single or multiple qubits.
- These gates are reversible.
- Example
  - $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Applying a Unitary Evolution

- For a state vector
  - $U|\psi\rangle$
- For a density matrix
  - $\rho = \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho U^\dagger$

State Fidelity

- Is the measure of closeness between two quantum states
- $F(\rho, \sigma) = \left\| \sqrt{\rho}\sqrt{\sigma} \right\|_1^2, \;\; 0 \leq F \leq 1$
  - Where F represents fidelity and ρ and σ represent either pure or mixed state.
- To measure the closeness between the noisy state and the ideal one, the fidelity will be reduced to

  $F(\rho, \sigma) = \langle \psi_\rho | \sigma | \psi_\rho \rangle$, where $\rho = |\psi_\rho\rangle\langle\psi_\rho|$
- If both states are pure, the fidelity will be $\left| \langle \psi_\rho | \psi_\sigma \rangle \right|^2$

Example

- Compute the fidelity between 2 pure orthogonal states $|0\rangle$ and $|1\rangle$
  - *Solution:*

$$F = |\langle 0|1\rangle|^2 = \left| \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right|^2 = 0$$

Example

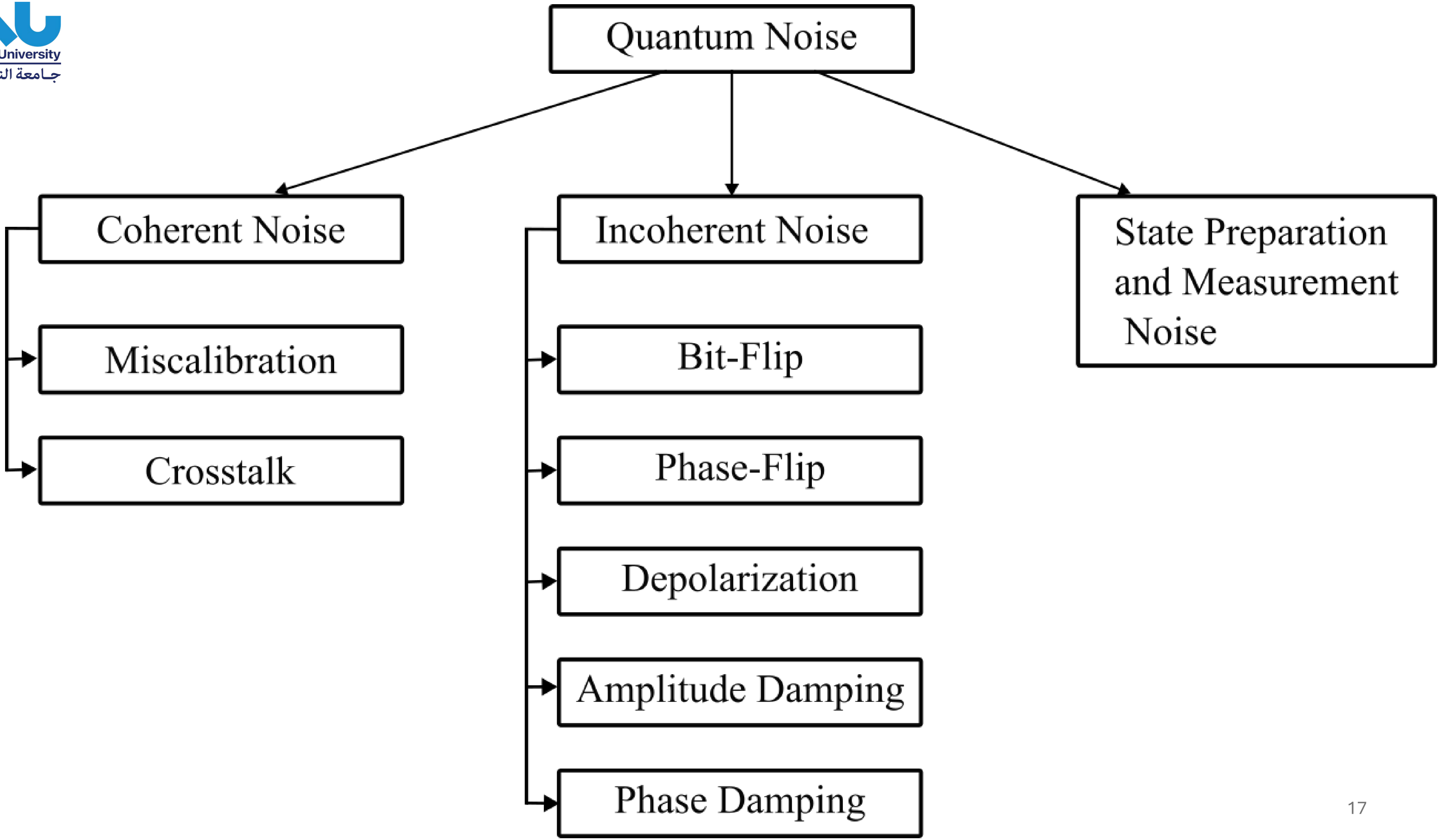
- Compute the fidelity between pure state $\psi_\rho = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and mixed state $\sigma = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$

Solution:

$$F(\rho, \sigma) = \langle \psi_\rho | \sigma | \psi_\rho \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} . \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

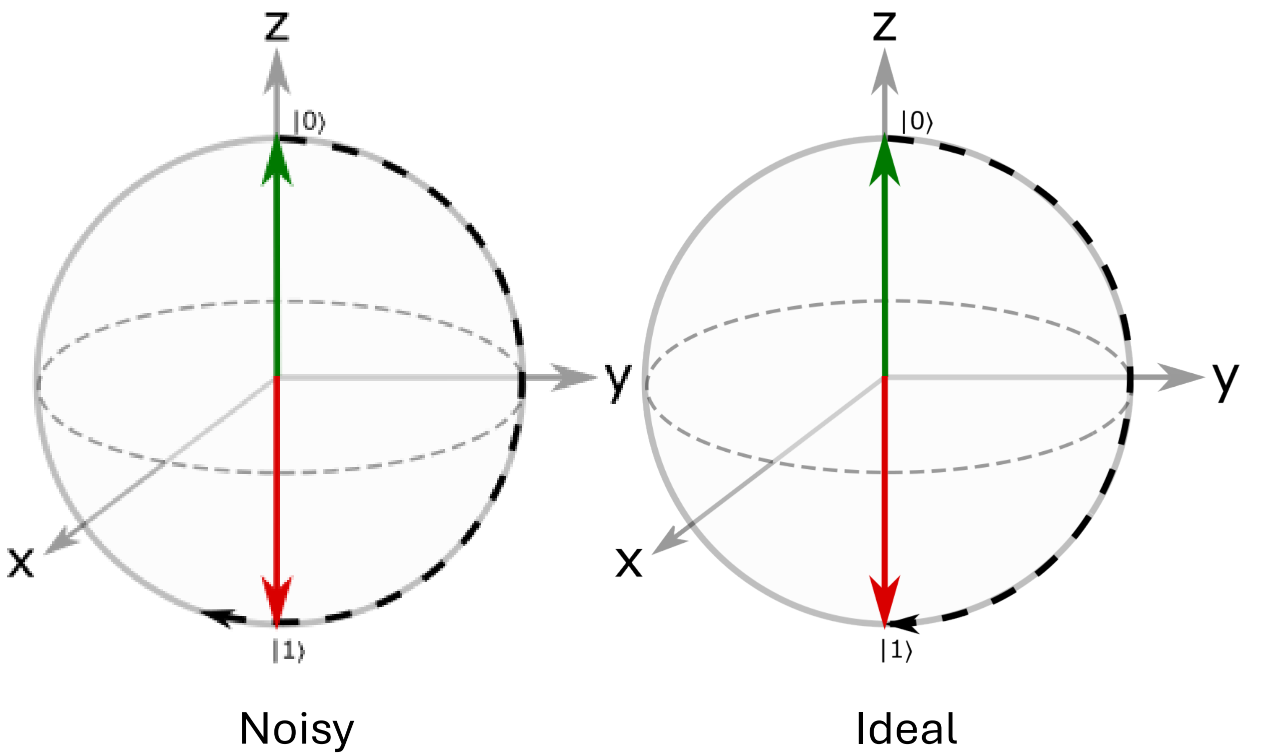
$$= 0.5$$

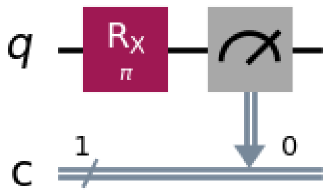
Types of Noise

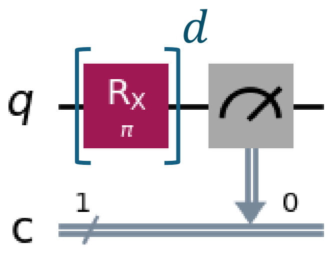

Quantum Noise
Coherent Noise
Miscalibration
Crosstalk
Incoherent Noise
Bit-Flip
Phase-Flip
Depolarization
Amplitude Damping
Phase Damping
State Preparation and Measurement Noise

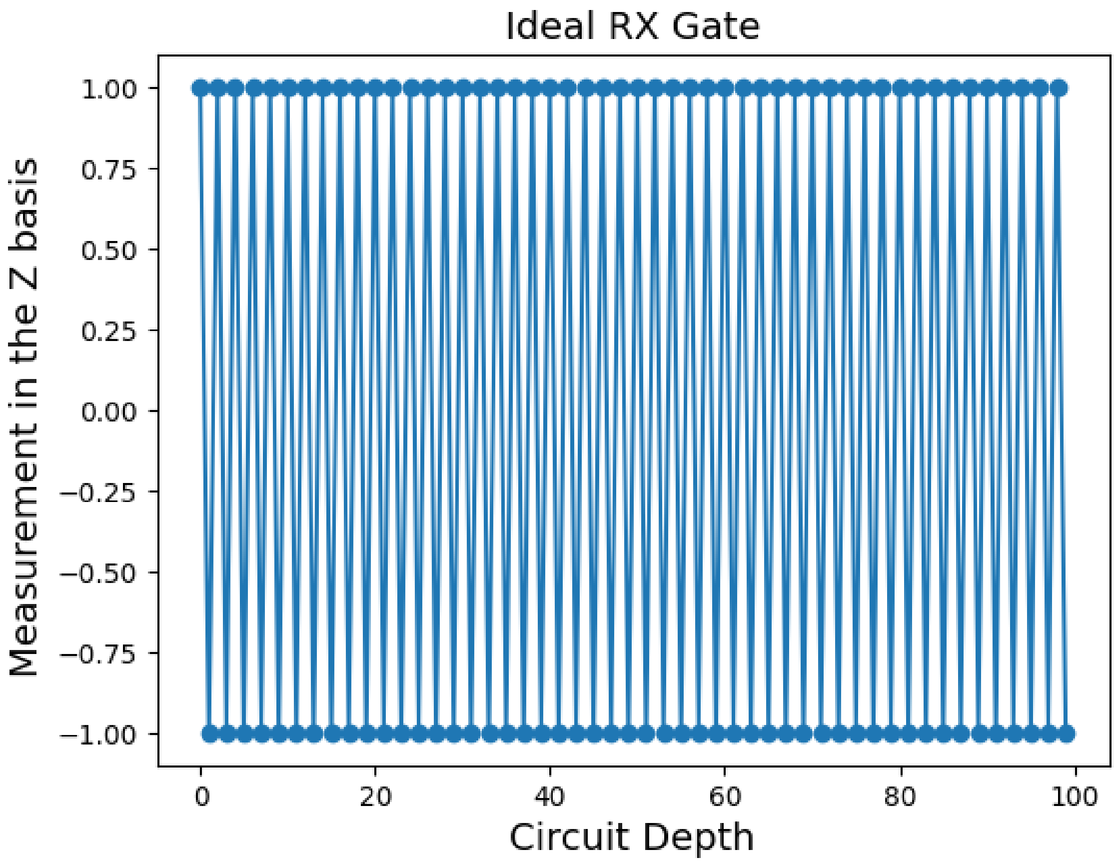
Coherent Errors

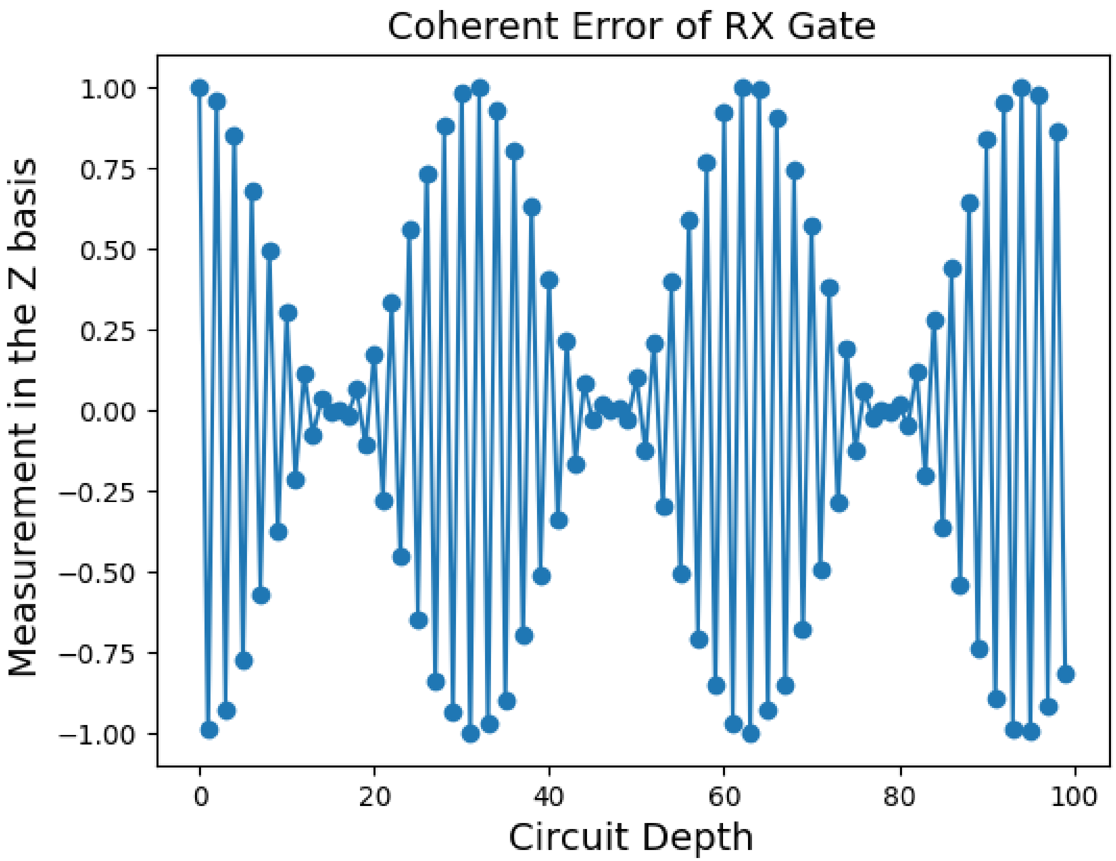
- Miscalibrated gates resulting in over or under rotations.
- Suppose, you have an X gate ($X = R_x(\pi)$),
- Due to noise, the state will end up $\tilde{X} = R_x(\pi + \epsilon)$, where ϵ is the additional error.








Ideal RX Gate
Measurement in the Z basis
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
20
40
60
80
100
Circuit Depth


Coherent Error of RX Gate
Measurement in the Z basis
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
20
40
60
80
100
Circuit Depth

Crosstalk Noise

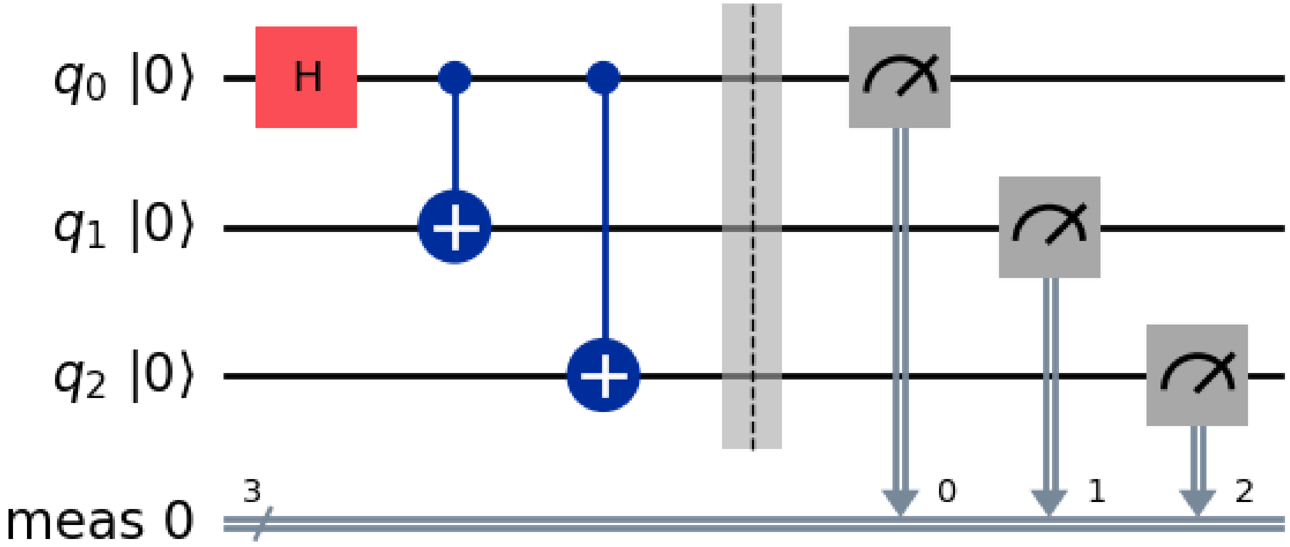
- Unwanted noise due to execution of multiple gates in parallel.
- Noise added to qubits other than those originally operated by the gate

Incoherent Errors

- Unwanted interactions with the environment resulting in a probability distribution of several pure states
- The state can be represented as a mixed state

  $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, where p_j represents the probability of being in the pure state $|\psi_j\rangle$.

Consider Simulating 3-qubit GHZ Circuit



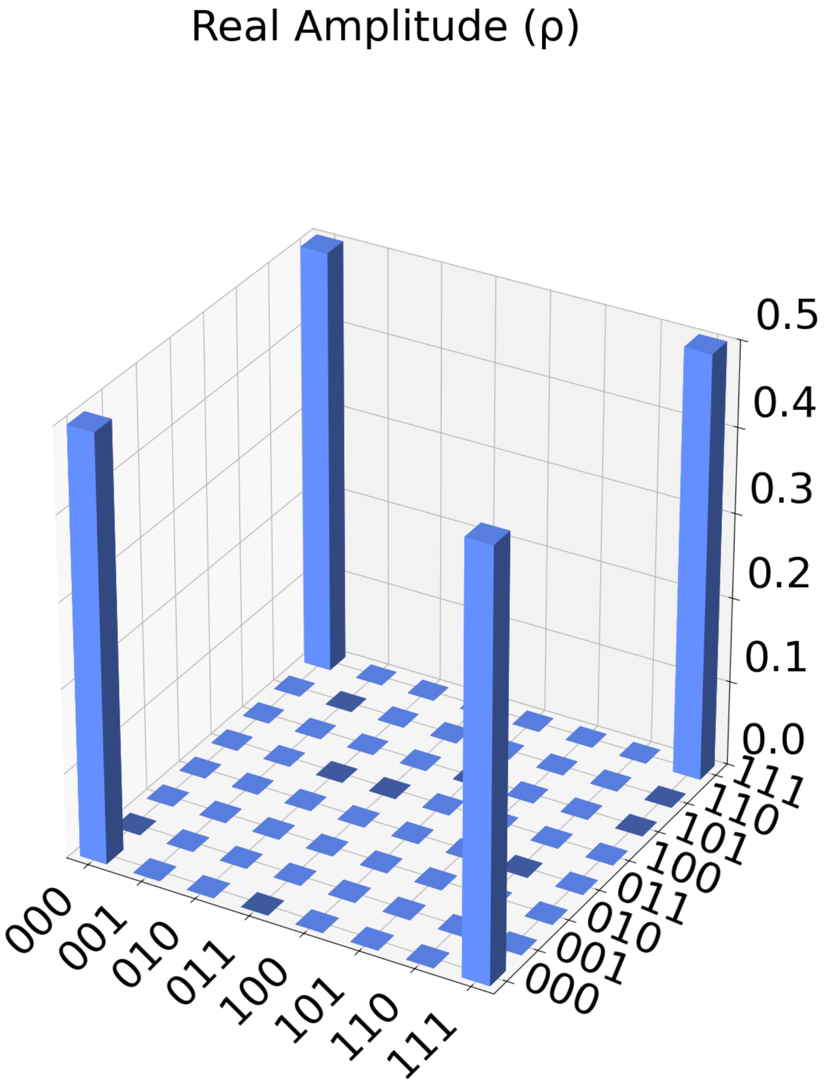
Ideally, the state before measurement is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

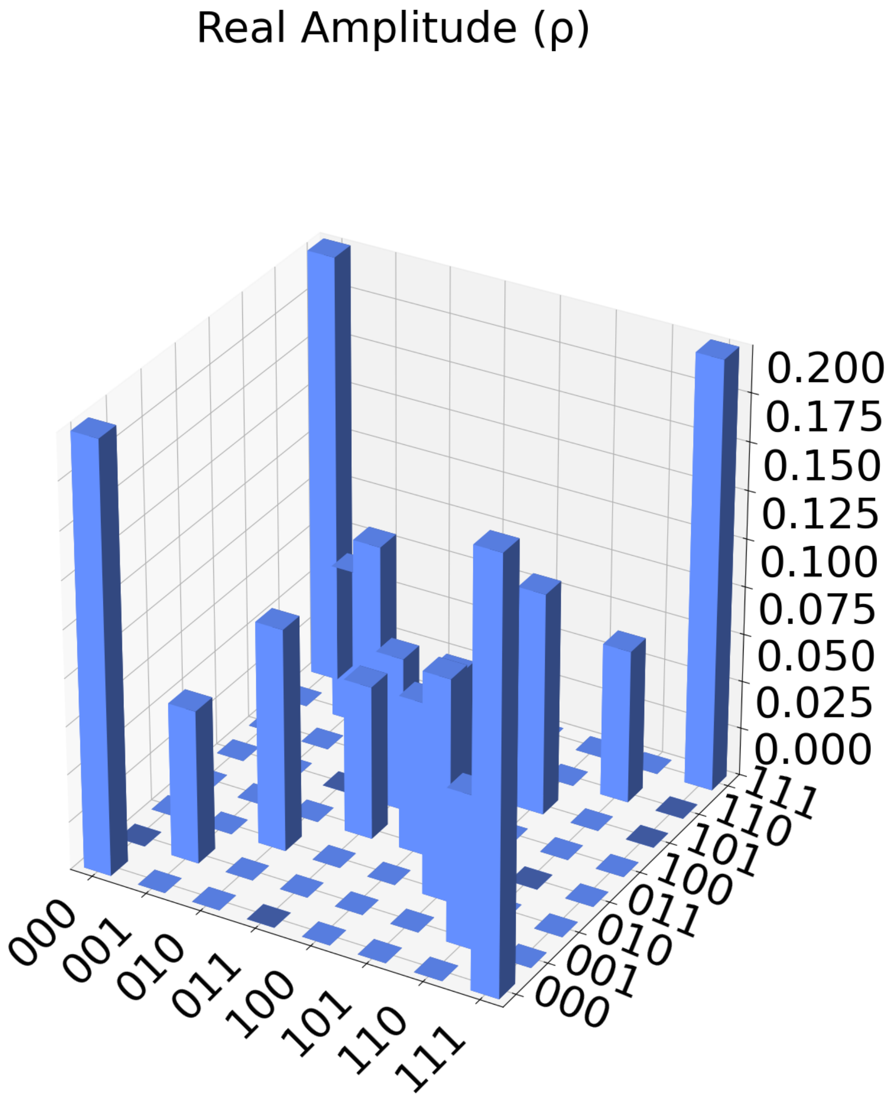
Bit Flip Error

- Flip the qubit state with probability p
- Can be applied after a certain unitary U
  - $\mathcal{E}(\rho) = (1-p)U\rho U^\dagger \;\; + p\,(XU\rho U^\dagger X)$

Ideal



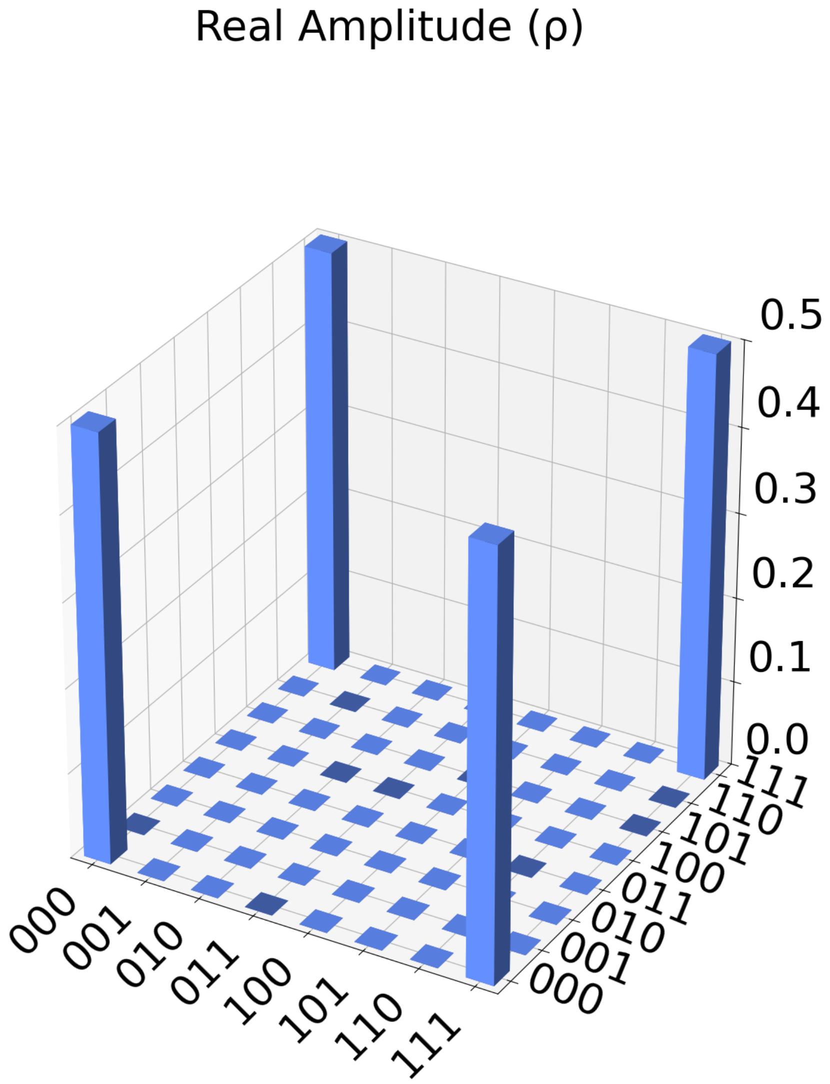
Bit Flip (p=0.2)



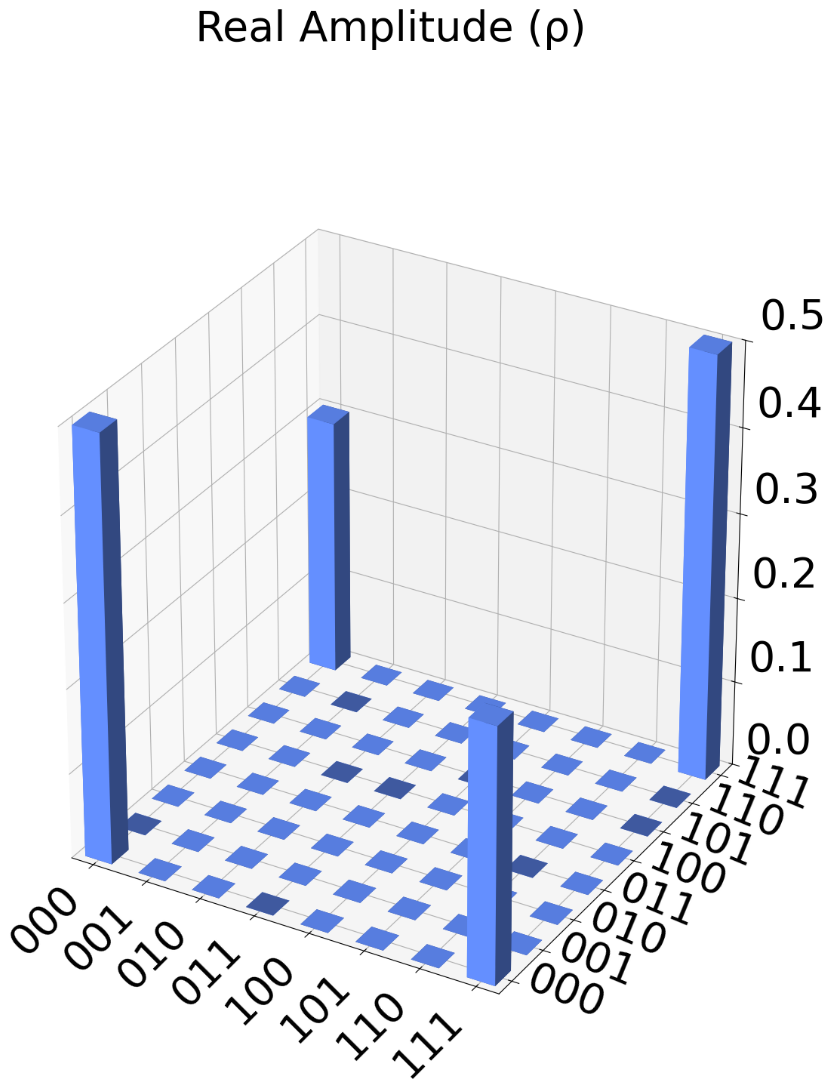
Phase Flip Error

- Flip the phase of a qubit state with probability p
- Can be applied after a certain unitary U
  - $\mathcal{E}(\rho) = (1-p)U\rho U^\dagger \ \ + p\,(ZU\rho U^\dagger Z^\dagger)$

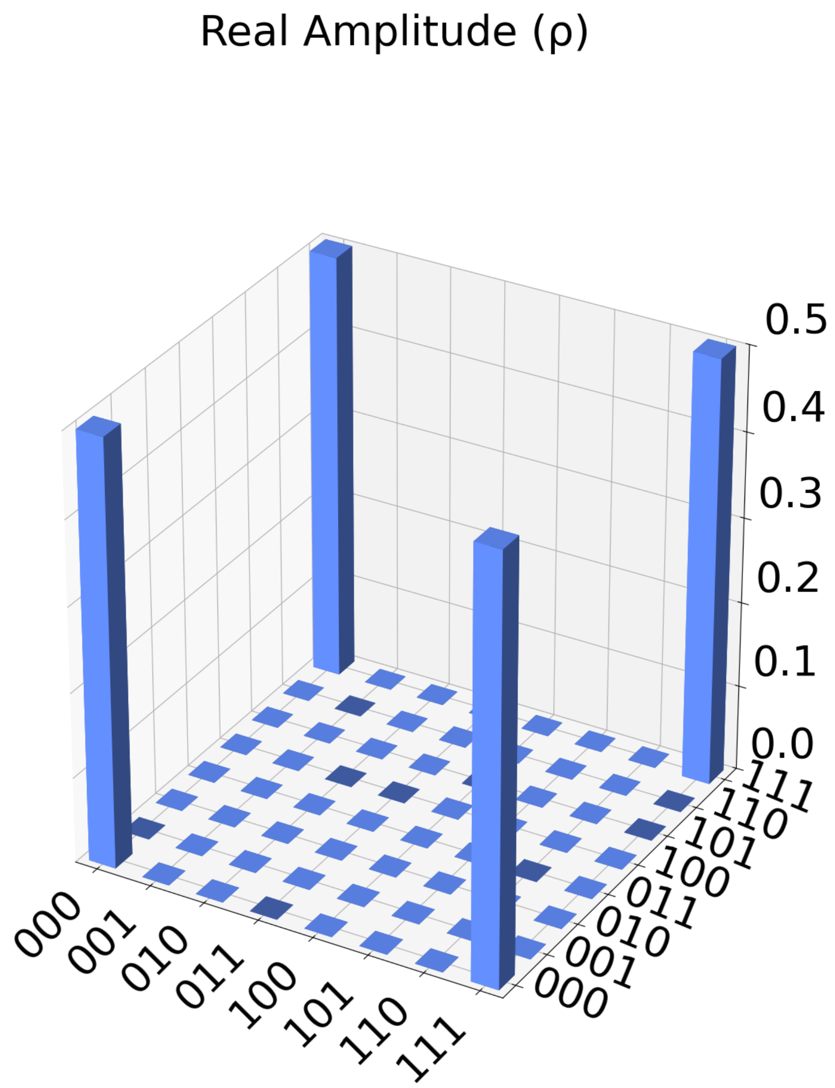
Ideal



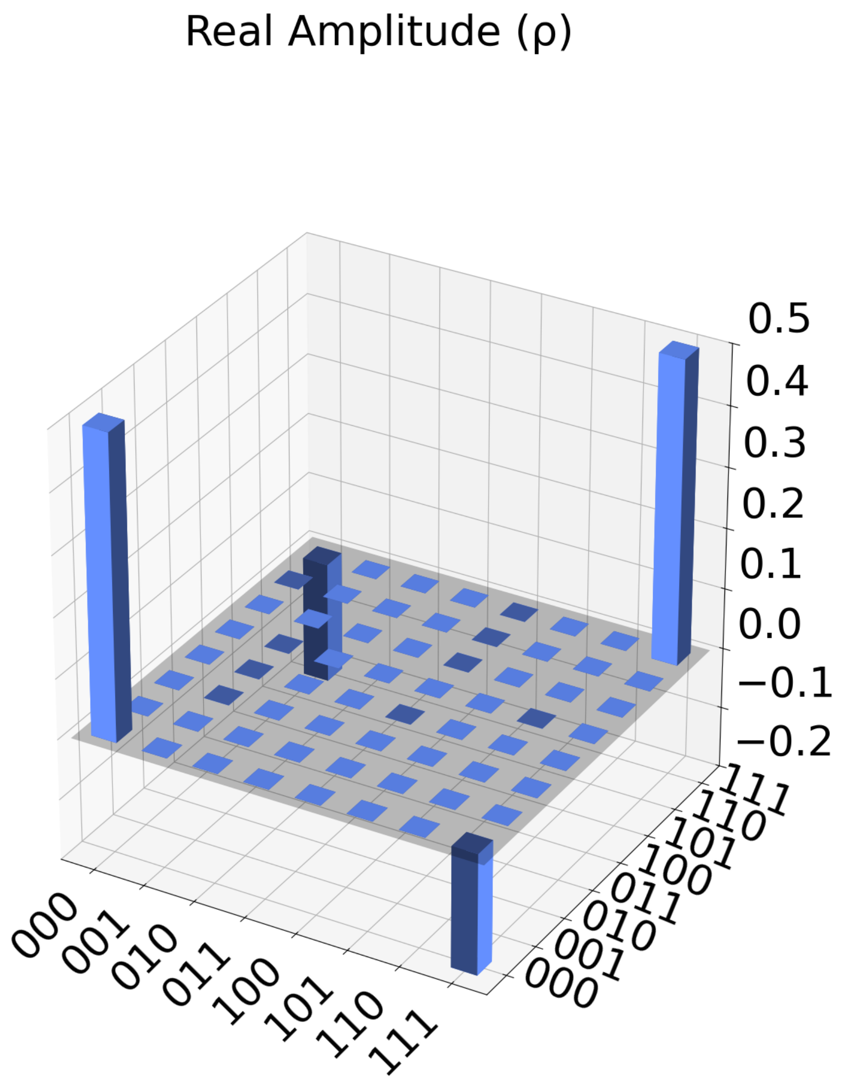
Phase Flip (p=0.2)



Ideal



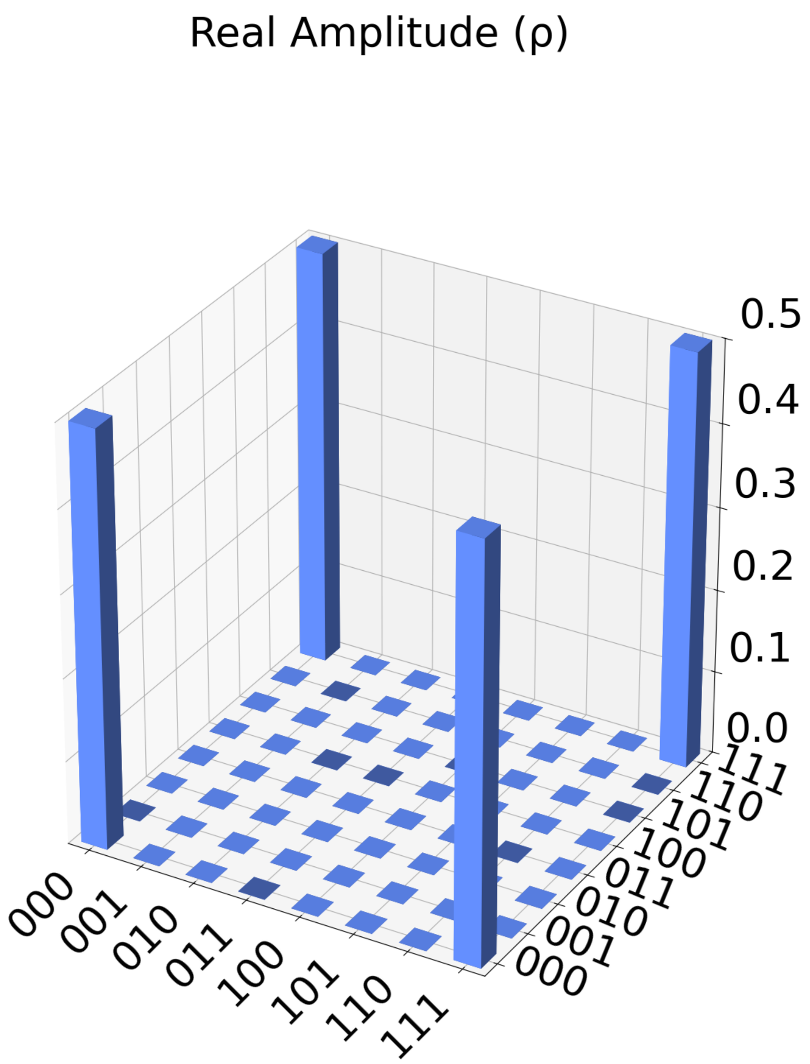
Phase Flip (p=0.7)



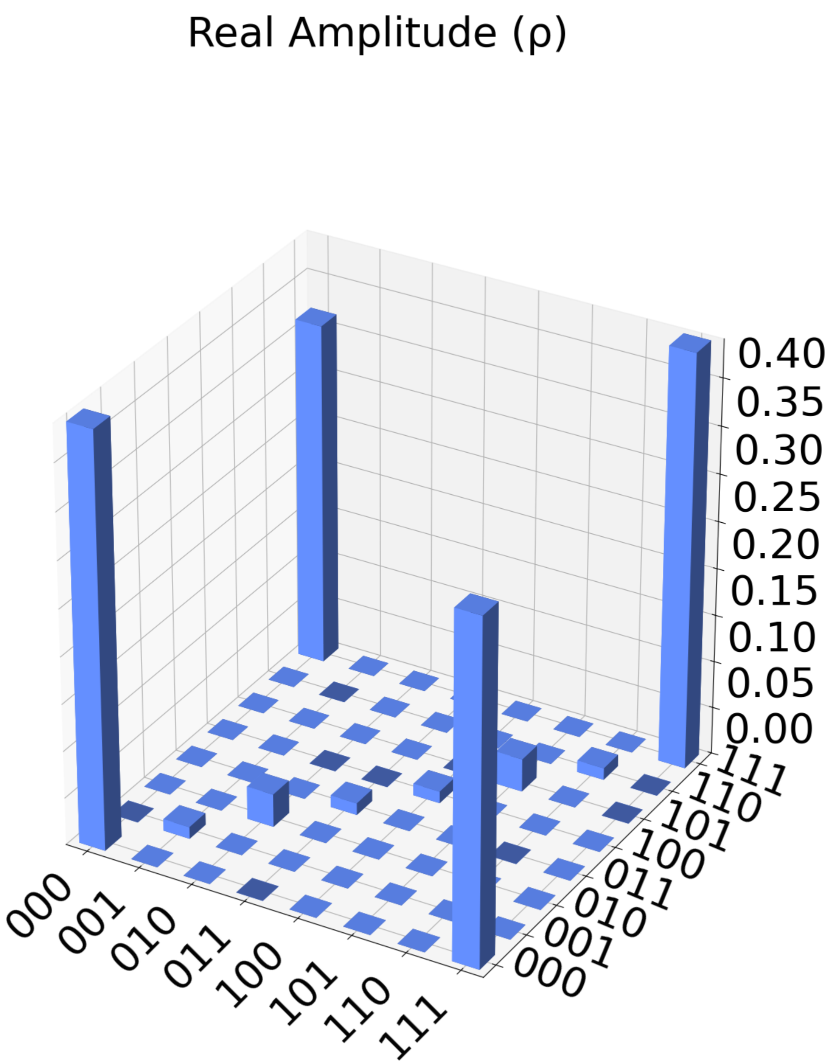
Depolarizing Error

- Due to imperfect gates
- A unitary U is correctly applied with probability $1 - p$ and followed by equally probable Pauli error after the gate with probability p
  - $\mathcal{E}(\rho) = (1-p)U\rho U^\dagger \ \ + \frac{p}{P_n}\sum_{P\in P_n} PU\rho U^\dagger P^\dagger$
- Consider the depolarization error on a single qubit
  - $\mathcal{E}(\rho) = (1-p)U\rho U^\dagger \ \ + \frac{p}{3}(XU\rho U^\dagger X^\dagger + YU\rho U^\dagger Y^\dagger + ZU\rho U^\dagger Z^\dagger)$

Ideal



Depolarization $p = 0.1$

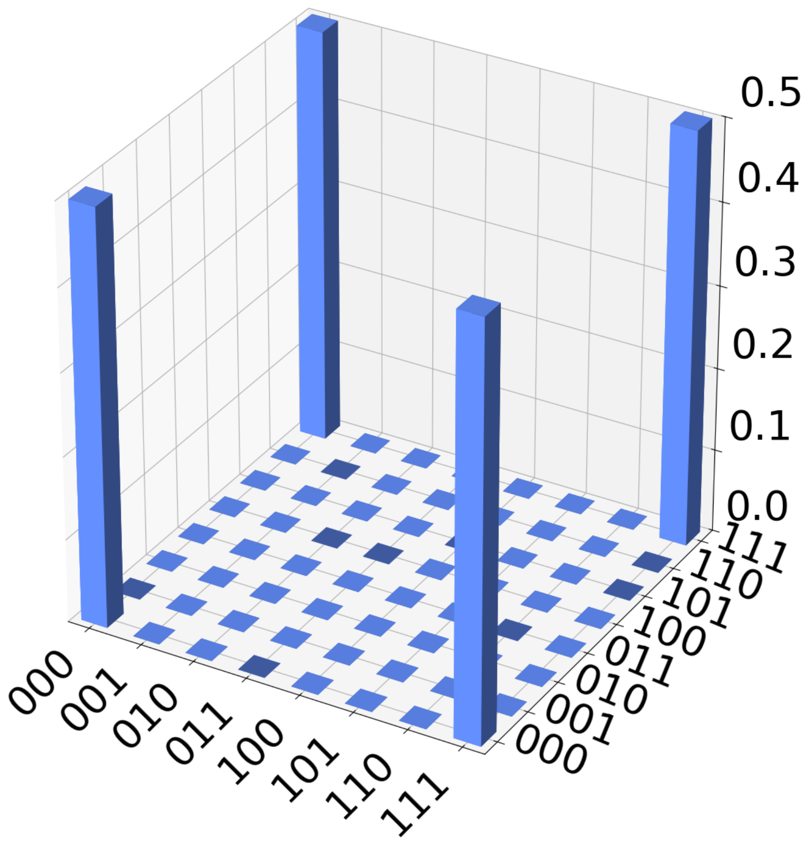


Amplitude Damping

- Energy loss of the excited state of a qubit in $|1\rangle$ to the ground state energy $|0\rangle$.
- $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$
- $\mathcal{E}_{AD}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$

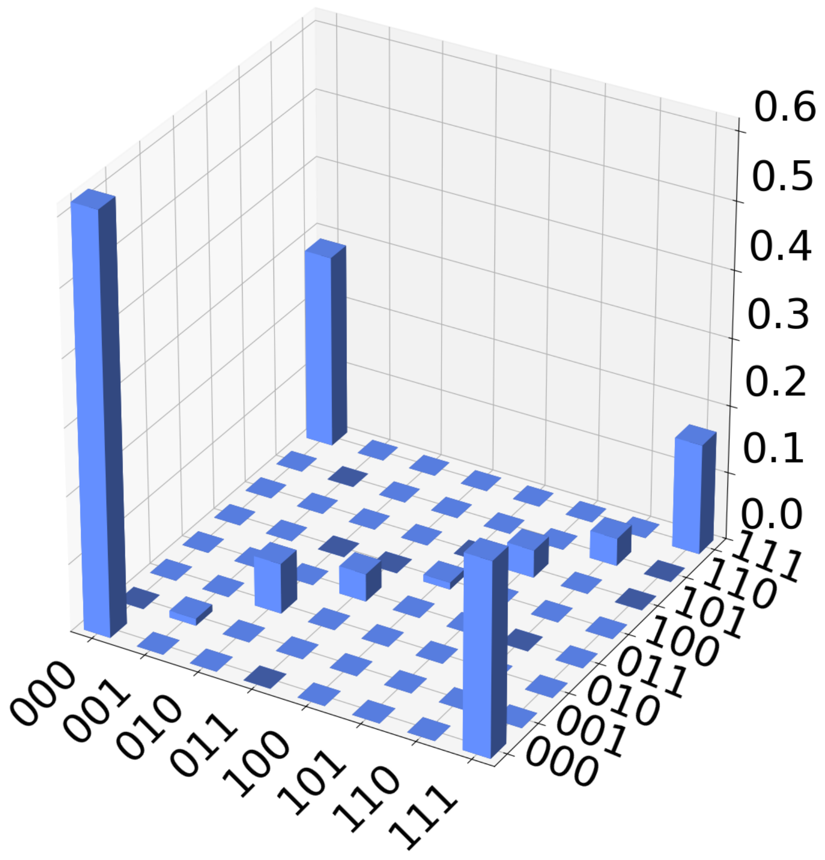
Ideal

Real Amplitude (ρ)



Amplitude Damping $p = 0.2$

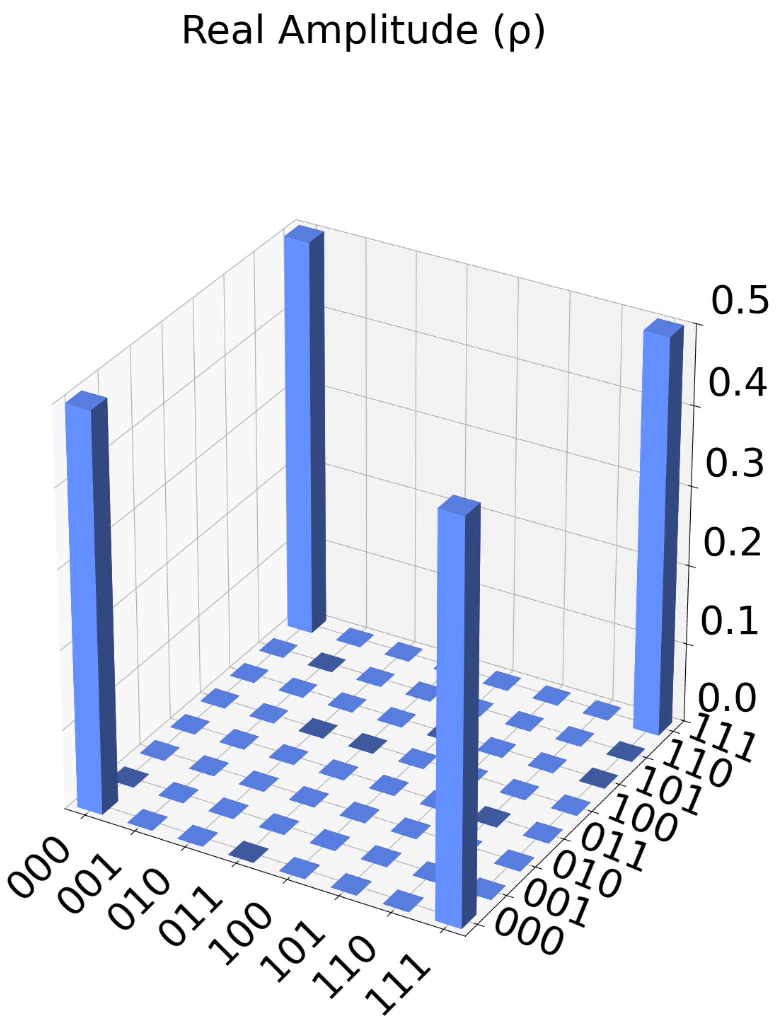
Real Amplitude (ρ)



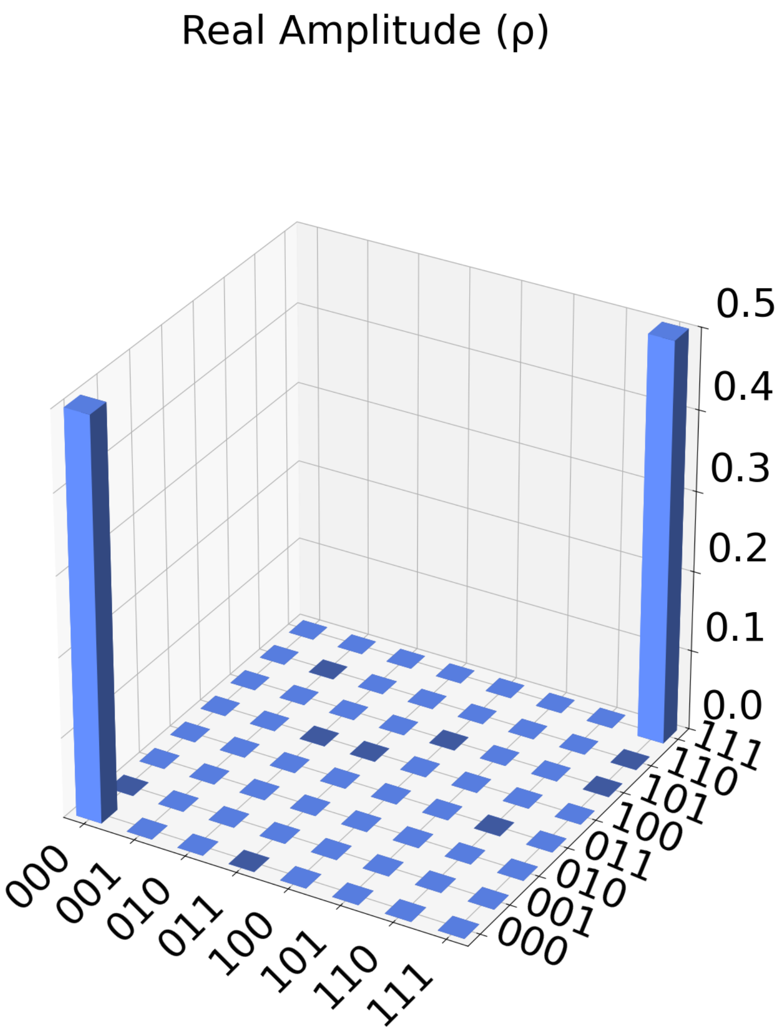
Phase Damping

- Decay of phase due to interactions with the environment.
- Pure state is transformed to mixed state.
- $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix}$
- $\mathcal{E}_{PD}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$

Ideal

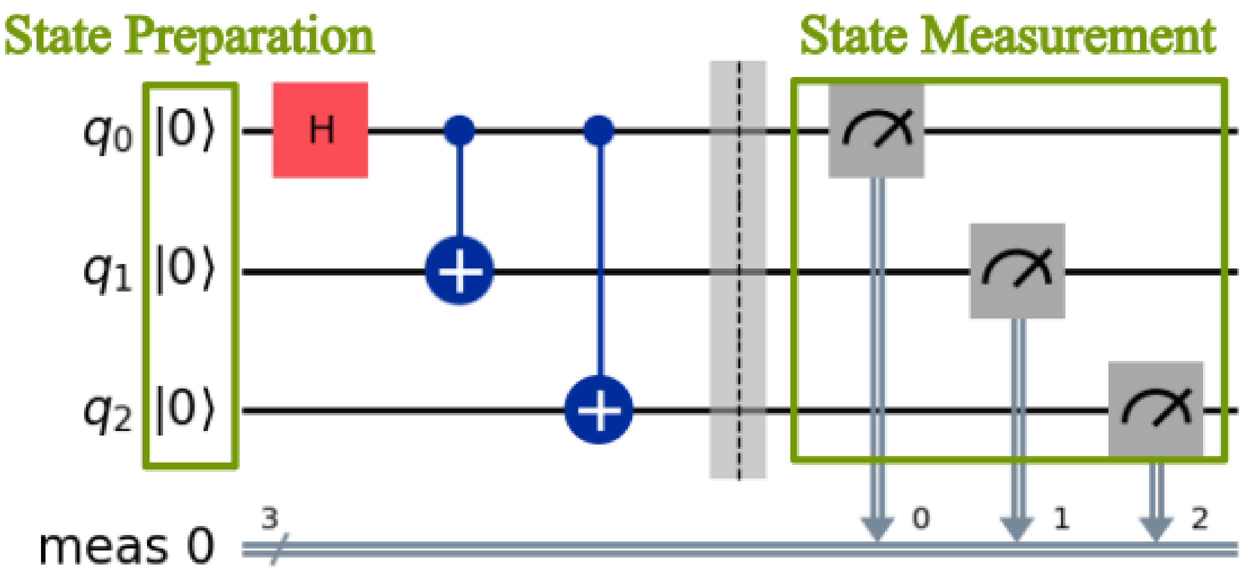


Phase Damping ($\gamma = 1$)



State Preparation and Measurement Errors (SPAM Errors)

- Errors in state initialization or preparation
  - Initialized $|0\rangle$ but ended up in $|1\rangle$
- Readout Errors in measurement
  - Should measure $|1\rangle$ but measured $|0\rangle$ instead.



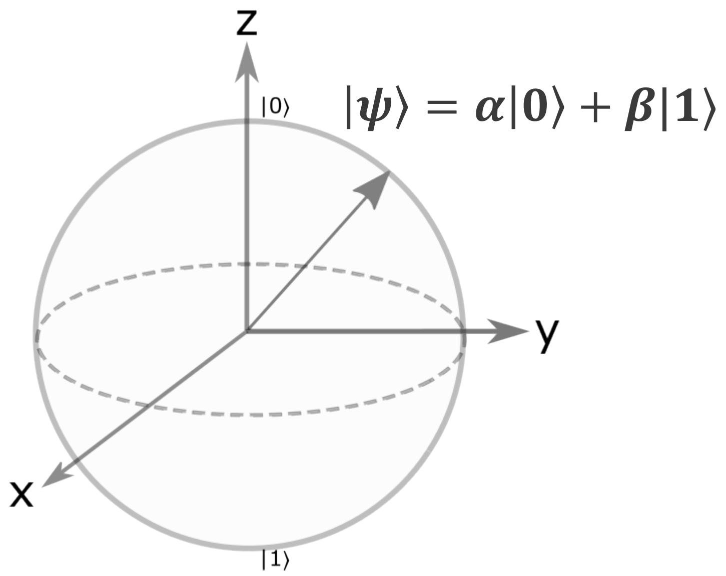
Noise Characterization

Simple Approach

- Multiply all success rates of gates in the circuit along with qubits' readout errors.
  - $P_{success} = \prod_{g \in G}(1 - \epsilon_g) \cdot \prod_{q \in Q}(1 - \epsilon_q)$
  - ϵ represents error, g represents gates and q represents qubits.

State Tomography

- a method to perform a series of projections to a circuit to reconstruct its state.
- For a single qubit state reconstruction, project on x, y and z axes.
- For a two-qubit state, project on $xx, xy, xz, \ldots, zz$ axes.
- For a n-qubit state, 3^n projections will be required.

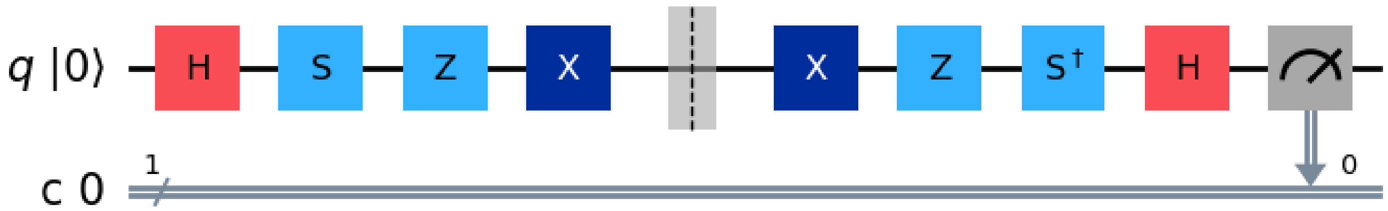


State Reconstruction steps

1. Generate a circuit
2. Calculate the ideal output state vector of this circuit
3. Choose the type of projectors (e.g., Pauli)
4. For a given circuit, the tomographic measurements are appended at the end of the replicas of the same circuit resulting in 3^n circuits.
5. Repeat the previous step multiple shots.
6. Reconstruct the density matrix ρ from these projective measurements using maximum likelihood technique.
7. To quantify the noise in the device, measure the fidelity between the ideal state vector and the reconstructed density matrix, $F(\rho, \sigma) = \langle \psi_\rho | \sigma | \psi_\rho \rangle$.

Randomized Benchmarking

- A scalable method to measure the Clifford gates fidelity
- Steps:
  - Initialize qubits to $|0\rangle$
  - Apply random sequences of Clifford gates of different lengths
  - For each random sequence, apply its inverse at the end of the circuit
  - Measure the circuit
- Ideally, the initial state $|0\rangle$ should be measured.
- Noise will cause the fidelity to decay as the sequence length increases.



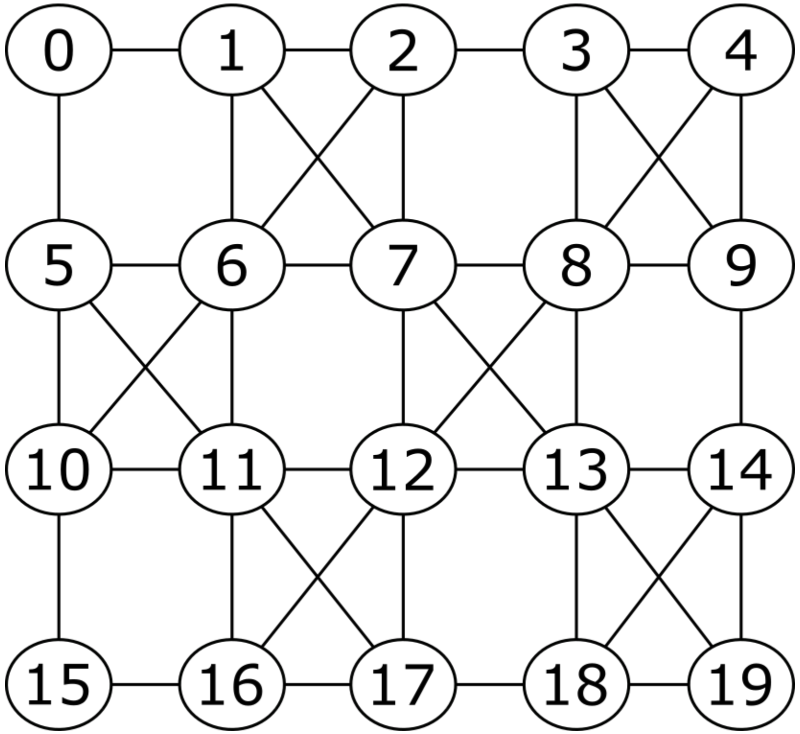
Noise Mitigation

Approaches

- Mapping Optimization
- Circuit Optimization
- Quantum Error Correction Codes
- Fault Tolerant Quantum Computing

Mapping Optimization

- Assign logical qubits to physical ones based on
  - Error rates
  - Hardware-Connectivity (reduce extra swap gates!!!)

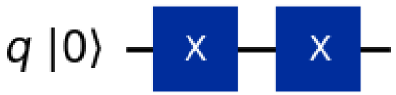


Hardware-Connectivity

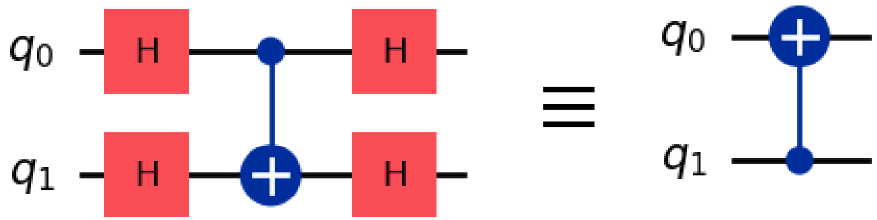
Circuit Optimization

- Gates Cancellation

$UU^{\dagger} = U^{\dagger}U = I$

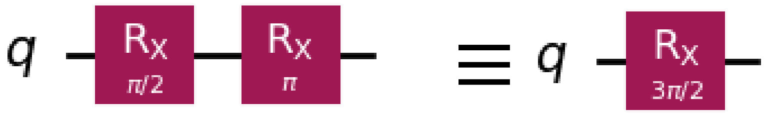


- Gates Replacement



- Gates Combination (ex. Combine 2 consecutive RX gates)

$RX(a)RX(b) = RX(a + b)$



Quantum Error Correction

Outline

- Repetition Codes
  - Bit-Flip code
  - Phase-Flip code
  - Or Both (Shor !!)
- Steane Code

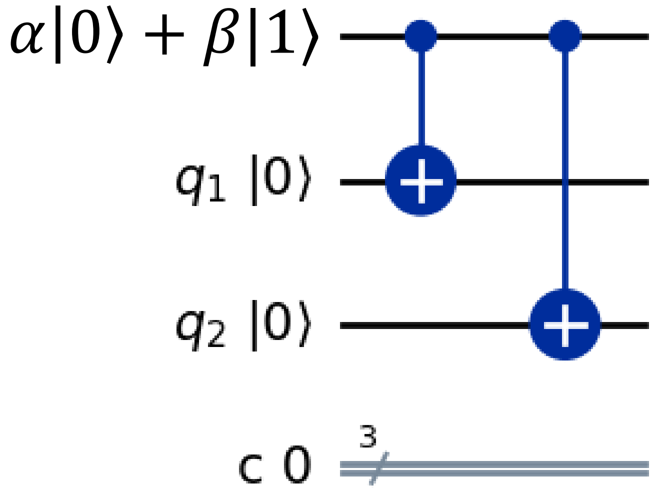
Classical Repetition codes

- Assume we want to send classical 0 over a channel
  - Send three consecutive 0s
  - Decode based on "**Majority Vote**"
    - Success : $(000,001,010,100) \to 0$
    - Failure if 2 or more bits flipped $(011,101,110,111) \to 1$
- Similarly, if we want to send classical 1

Three-qubit bit-flip code

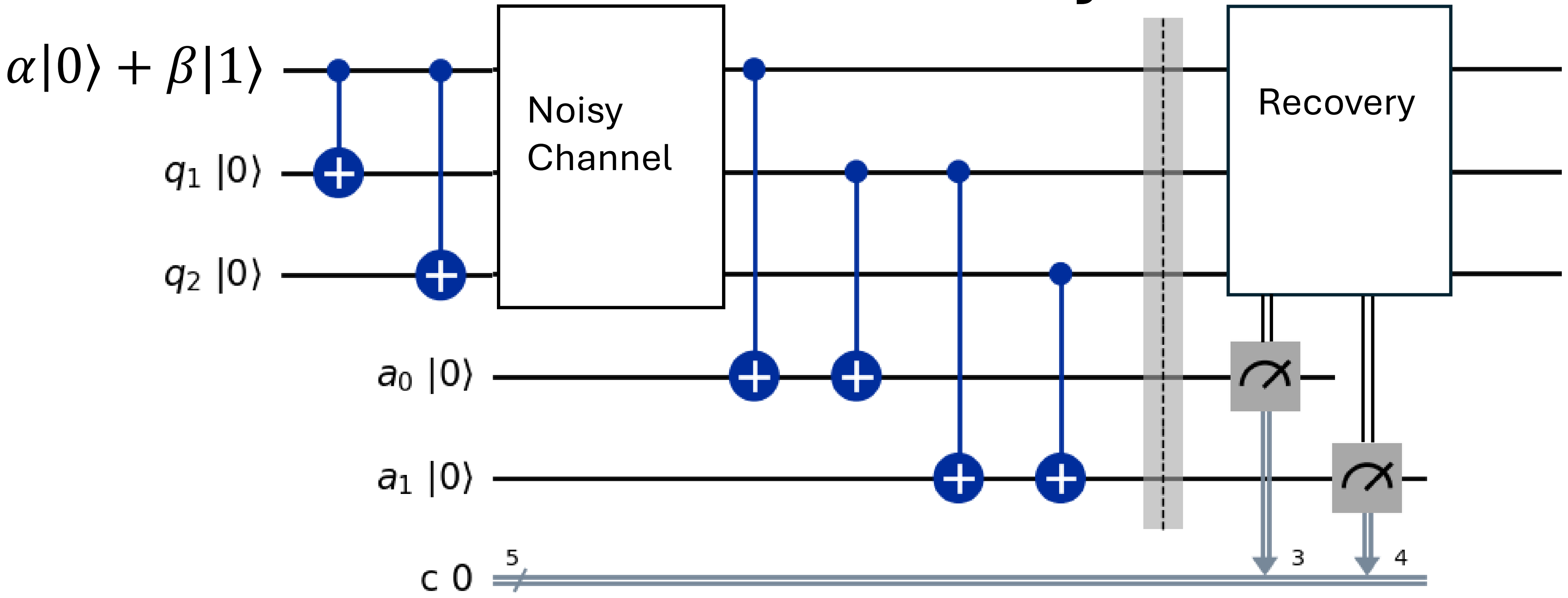
- Encode each qubit using three qubits:
  - $|0\rangle \rightarrow |000\rangle$
  - $|1\rangle \rightarrow |111\rangle$
- Can correct at most 1 error
- Error can be detected using parity checks.

Encoding Circuit



$$(\alpha|0\rangle + \beta|1\rangle)|00\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$$

Error Detection and Recovery

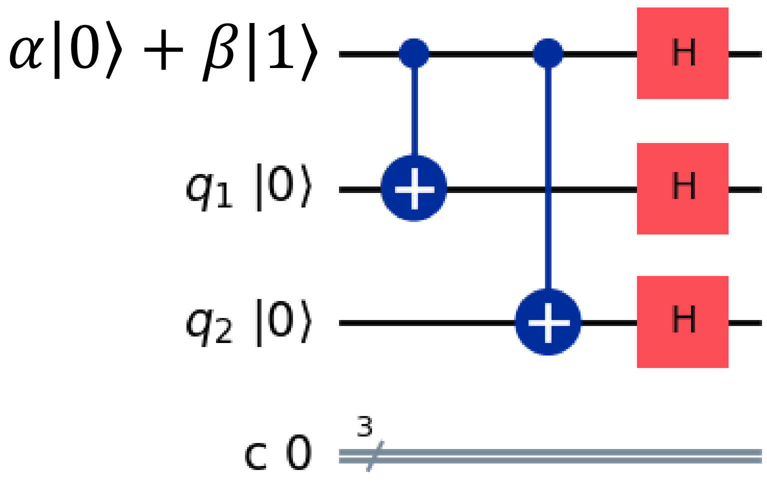


$|\psi\rangle$ After Noisy Channel	M_{a_0}	M_{a_1}	Recovery	$|\psi\rangle$ After Recovery
$\alpha|000\rangle + \beta|111\rangle$	0	0	$I \otimes I \otimes I$	$\alpha|000\rangle + \beta|111\rangle$
$\alpha|001\rangle + \beta|110\rangle$	0	1	$I \otimes I \otimes X$	$\alpha|000\rangle + \beta|111\rangle$
$\alpha|100\rangle + \beta|011\rangle$	1	0	$X \otimes I \otimes I$	$\alpha|000\rangle + \beta|111\rangle$
$\alpha|010\rangle + \beta|101\rangle$	1	1	$I \otimes X \otimes I$	$\alpha|000\rangle + \beta|111\rangle$

Three-qubit phase-flip code

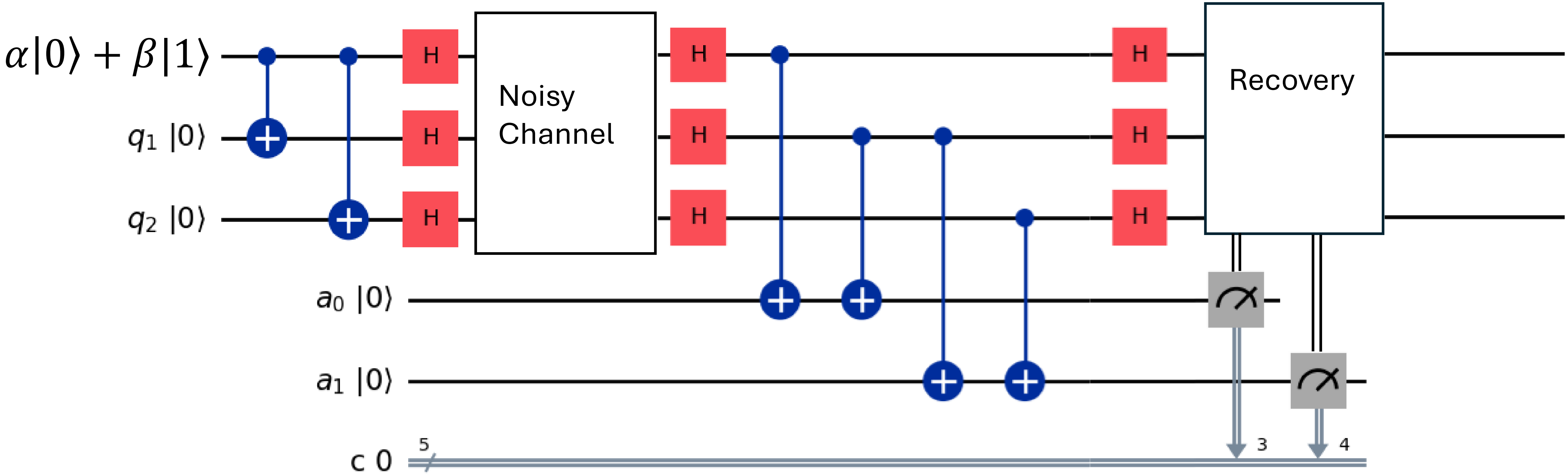
- Encode each qubit using three qubits:
  - $|0\rangle \rightarrow |++ +\rangle$
  - $|1\rangle \rightarrow |-- -\rangle$
- Can correct at most 1 error
- Error can be detected using parity checks.

Encoding Circuit



$$(\alpha|0\rangle + \beta|1\rangle)|00\rangle \rightarrow \alpha|+++\rangle + \beta|---\rangle$$

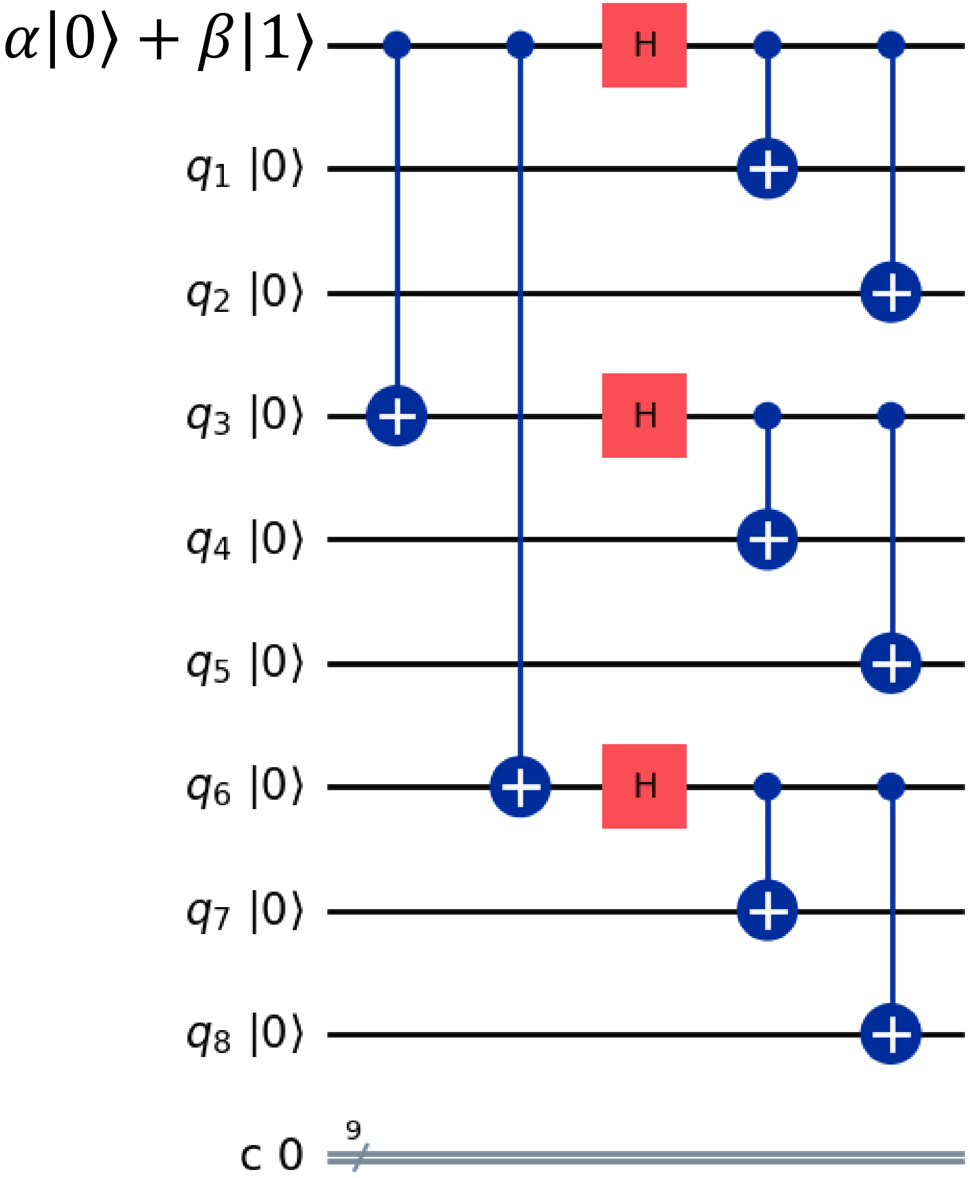
Error Detection and Recovery



$\|\psi\rangle$ After Noisy Channel	M_{a_0}	M_{a_1}	Recovery	$\|\psi\rangle$ After Recovery
$\alpha\|{+}{+}{+}\rangle + \beta\|{-}{-}{-}\rangle$	0	0	$I \otimes I \otimes I$	$\alpha\|{+}{+}{+}\rangle + \beta\|{-}{-}{-}\rangle$
$\alpha\|{+}{+}{-}\rangle + \beta\|{-}{-}{+}\rangle$	0	1	$I \otimes I \otimes Z$	$\alpha\|{+}{+}{+}\rangle + \beta\|{-}{-}{-}\rangle$
$\alpha\|{-}{+}{+}\rangle + \beta\|{+}{-}{-}\rangle$	1	0	$Z \otimes I \otimes I$	$\alpha\|{+}{+}{+}\rangle + \beta\|{-}{-}{-}\rangle$
$\alpha\|{+}{-}{+}\rangle + \beta\|{-}{+}{-}\rangle$	1	1	$I \otimes Z \otimes I$	$\alpha\|{+}{+}{+}\rangle + \beta\|{-}{-}{-}\rangle$

Shor Code

- Introduced by Peter Shor in 1995.
- Combines both bit-flip and phase-flip repetition codes
- Each logical qubit is encoded to 9 physical qubits



Steane Code

- Encodes a logical qubit using 7 qubits (more compact than Shor!!)
- Based on the hamming code (7,4), where a 4-bit data word is encoded to 7 bits.
- $|0_L\rangle = \frac{1}{\sqrt{8}}(|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle$
  $+|0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle)$
- $|1_L\rangle = \frac{1}{\sqrt{8}}(|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle$
  $+|1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle)$

Other Correction Codes

- Surface codes
- Color codes
- LDPC codes

Fault Tolerant Quantum Computing

Definition

- The ability to apply transformations to the quantum state even in the presence of noise.

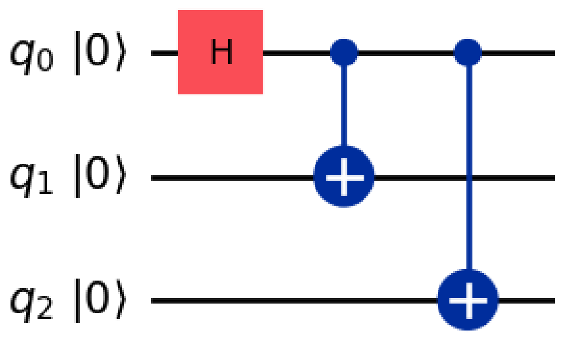
Fault Tolerance Properties

- The error propagates to at most one qubit in each block
- To achieve full fault tolerance, each element in the circuit should be fault tolerant (state preparation, gates, error correction, measurement)
- The physical error rate should be lower than the code threshold in order not to introduce additional noise

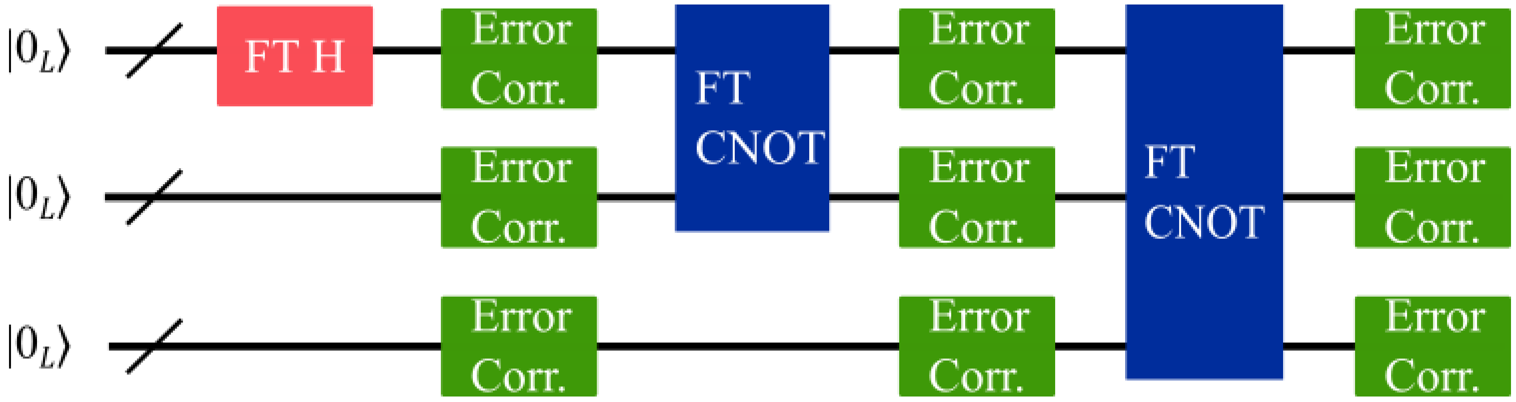
Steps to achieve fault tolerance

- Choose an error correction code
- Transform original circuit into its encoded version
- Apply gates in the original circuit transversally or use magic state injection
- Correct errors on regular intervals

Fault Tolerant Circuit Transformation



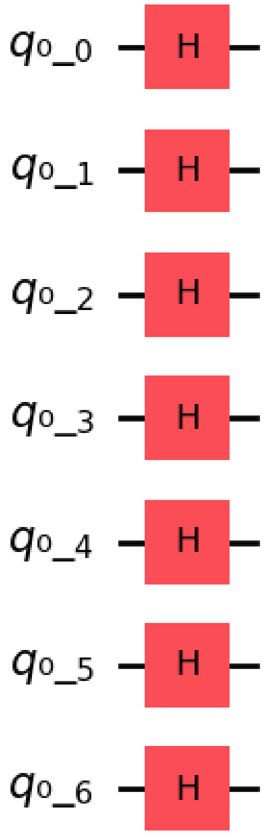
Original Circuit



Fault Tolerant Circuit

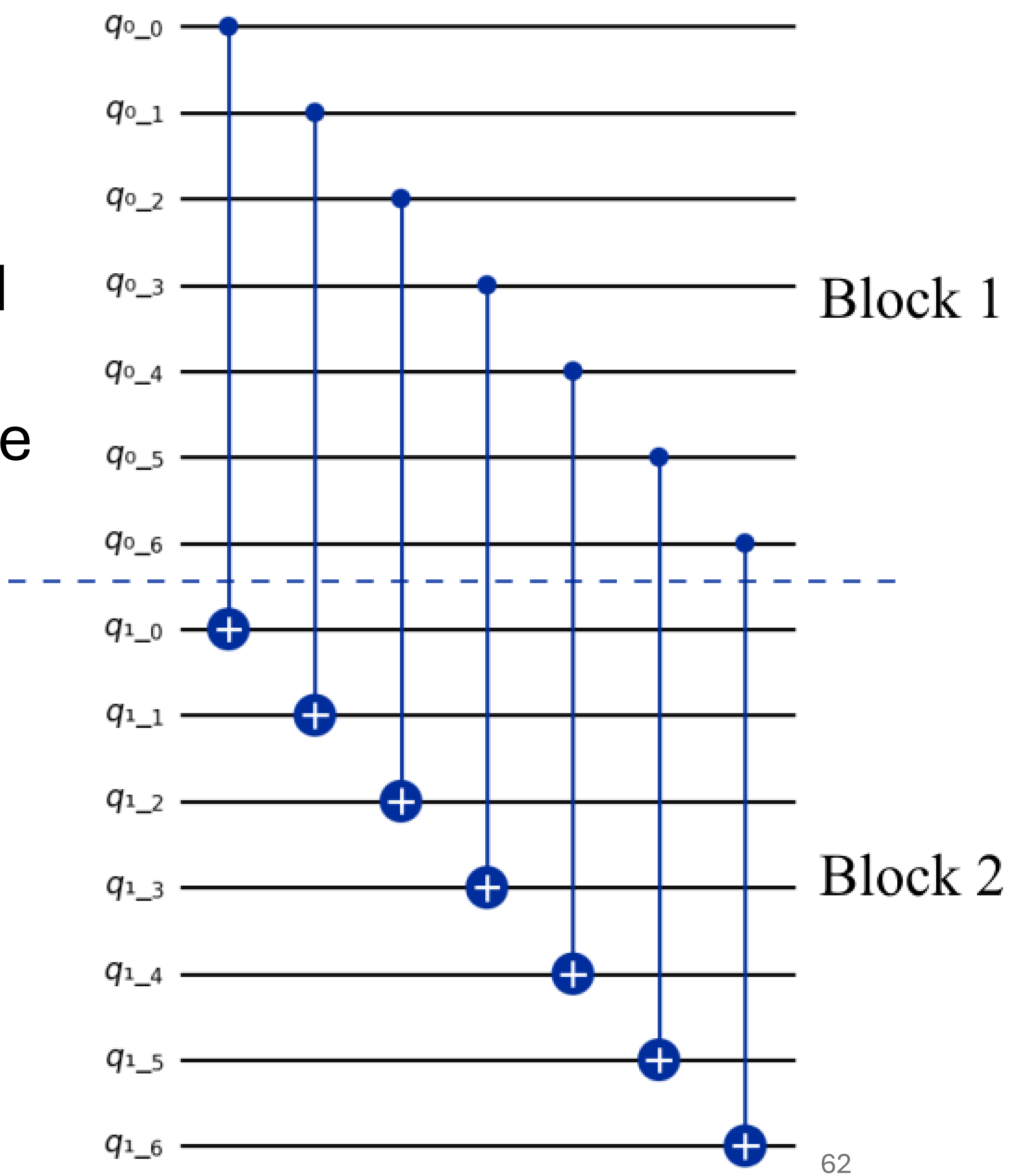
Implementing H Gate

- Hadamard gate (H) can be implemented transversally on each qubit of the 7-qubit Steane encoding.
- If an error occurred, it won't propagate to the rest of qubits.



Implementing CNOT Gate

- CNOT gates can also be implemented transversally between a qubit in the first block with its corresponding in the second block
- Error will at most propagate to one qubit from each block



Other Gates

- Not all gates can be implemented transversally (example T Gate in Steane code) by any error correction code according to Eastin-Knill theorem.
- Use gate teleportation (magic state injection)

Future

- IBM Quantum Starling, a large-scale fault tolerant quantum computer, will be introduced by 2029 capable of executing 100 million gates on 200 logical qubits.

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Thank you

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